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DIMENSIONAL ANALYSIS AND EQUATION FOR AXIAL HEAT FLOW OF GORTER–MELLINK CONVECTION (He II)*

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Abstract—Dimensional analysis of the heat transport through He II-filled ducts is carried out on the basis of Landau's two fluid equations for the limit of normal fluid depletion. Resulting functional relations between relevant groups of variables for Newtonian fluid behavior are extended up to the lambda point (T_λ) using the temperature dependence of the effective "turbulent" viscosity. The resulting convection equation is in good agreement with data of many authors and our own results from about 1.2 K to T_λ and for diameters of the order 10^{-2} to 1 cm.

NOMENCLATURE

A_{GM}, turbulent reciprocal viscosity [m s/kg];
D, hydraulic diameter [m];
j, mass flux density [kg/m^2 s];
K_{GM}, constant;
L_c, characteristic length [m];
N_i, dimensionless numbers ($i = 1, 2, 3, q, P_T$);
P_T, thermomechanical pressure [N/m^2];
q, heat flux density [W/m^2];
S, entropy [J/K kg];
t, time [s];
T, temperature [K];
v, velocity [m/s];
w, relative velocity [m/s].

Greek symbols

η, shear viscosity [kg/m s];
ρ, density [kg/m^3].

Subscripts

n, normal fluid;
s, superfluid;
λ, lambda point.

1. INTRODUCTION

RECENT applications of He II as a coolant for superconducting energy storage [1] and in space systems [2] have stimulated considerable interest in the superfluid thermohydrodynamics. He II permits unusual thermomechanical convection modes [3] at temperatures useful for optimum parameters of the superconducting state (e.g. of the most frequently used alloy Nib–Ti). Therefore we have extended investigations of He II transport limits [4] and post-transition studies after crossing of the lambda curve [5] to the regime of Gorter–Mellink convection [6]. This case is characterized by axial transport of thermal energy through a He II-filled duct with insulated walls which is heated at one end, and cooled at the other end. Because of a special, unknown transport property (A_{GM}), various theoretical approaches have received attention [7]. Improvements of this state of knowledge appear to be suggested by recent better understanding of the phenomenological continuum description of He II [8]. Therefore, it is the purpose of the present paper to propose a convection equation based on dimensional analysis of the ideal Landau equations [9, 10], extended to include dissipation of energy during real non-conservative flow of heat.

First, we consider in Section 2 the extended Landau equations of the two-fluid model approach in the limit of normal fluid depletion. This permits simplifications for the formulation of dimensionless groups of variables. Subsequently we turn in Section 3 to the Gorter–Mellink transport property and in particular to its relationship to the preceding section. This leads to a surprisingly simple function for A_{GM}. Finally, contact between the dimensionless frame of governing variables and experimental data is established in Section 4. The resulting convection equation covers data in a wide range of the dimensionless driving potential.

2. NORMAL FLUID DEPLETION LIMIT

According to the two-fluid model both normal and superfluid contribute to the density

$$\rho_s/\rho + \rho_n/\rho = 1. \qquad (1)$$

At thermodynamic equilibrium, the contributions on the LHS of equation (1) are unique functions of the temperature. At low T, we have $\rho_n/\rho \ll 1$, i.e. little normal fluid, and consequently a small entropy. In this limit a dimensional analysis is simplified because the superfluid density ratio $(\rho_s/\rho) \approx 1$. Therefore it is expected that superfluid constraints may not show up explicitly. A simplified description of convection appears to be possible. Consider the Landau equations [9, 10] modified for the special case of heat flow. Its non-linear two-fluid model thermohydrodynamics has

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been treated for undamped oscillations without dissipation [11]. Friction, however, is important for the present transport of heat. The simplest non-conservative case is viscous dissipation by the normal fluid whose friction is proportional to $\eta_n \nabla^2 \mathbf{w}$ (without additional friction forces). Adopting this term we rewrite the modified Landau equations as follows:

Mass flux density constraint:

$$\mathbf{j} = \rho \mathbf{v} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0. \qquad (2)$$

Axial steady heat flow (expressed as relative flow velocity $\mathbf{w} = (\mathbf{v}_n - \mathbf{v}_s)$):

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\{q/(\rho_s S T)\} = 0. \qquad (3)$$

Thermomechanical pressure gradient (from irreversible thermodynamics [12]):

$$\nabla P_T = \rho S \nabla T. \qquad (4)$$

Equation of relative flow ("counterflow"):

$$D\mathbf{w}/dt = \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\,\mathrm{grad})\mathbf{w}$$
$$= -(\rho/\rho_n) S \,\mathrm{grad}\, T + (\eta_n/\rho)\nabla^2 \mathbf{w}. \qquad (5)$$

Using the standard techniques of similarity studies we arrive at the dimensionless numbers of Table 1 along with the ratio (ρ_s/ρ_n) for steady flow.

Concerning entry N_1 in Table 1 we note that equations (2) and (3) imply

$$\rho \bar{v}_n = \rho_s \bar{w} = \rho_s \bar{v}_s(\rho/\rho_n). \qquad (6)$$

Further, early approaches made use of the "Leiden Reynolds number" [13] $N_1 = \rho \bar{v}_n L_c/\eta_n$. In view of equations (3) and (6) we express subsequently the dimensionless counterflow rate ($L_c = D$) as

$$N_1 = \bar{w} D \rho/\eta_n = \frac{\bar{q} D}{\eta_n S T}(\rho/\rho_s)$$
$$= N_q \cdot (\rho/\rho_s). \qquad (7)$$

A dimensionless pressure gradient without flow rate term is obtained as the product $N_1 \cdot N_3$ of Table 1.

$$N_1 N_3 = L_c^3 |\nabla P_T| (\rho/\eta_n^2)(\rho/\rho_n)$$
$$= N \nabla_{PT}(\rho/\rho_n). \qquad (8)$$

Gorter–Mellink transport is realized at relatively large $|\nabla T|$ with duct diameters above the order of magnitude 10^{-3} cm. If the normal fluid behaves indeed as a Newtonian system, a simple functional relation between (7) and (8) should result. However, for an extended temperature range further information is needed. Therefore we turn to the reciprocal turbulent viscosity A_{GM}, known as Gorter–Mellink mutual friction parameter.

Table 1. Dimensionless numbers

N_i	Comments
$N_1 = \bar{w} L_c \rho/\eta_n$	= "Leiden Reynolds number" for $\bar{j} = 0$ and $\rho_s \approx \rho$
$N_2 = \|\nabla P_T\| L_c/\rho_n \bar{w}^2$	dimensionless thermomechanical pressure gradient
$N_3 = \dfrac{\|\nabla P_T\| L_c^2 \rho/\rho_n}{\bar{w}\eta_n}$	represents transport resistance (~ratio of pressure difference to transport rate)

3. GORTER–MELLINK TRANSPORT PROPERTY

In the original Gorter–Mellink result A_{GM} is an unknown function obtained from experimental data for $\bar{q} = \bar{q}(\mathrm{grad}\, T)$

$$\bar{q} = \rho_s S T \left\{\frac{S|\nabla T|}{A_{GM}\rho_n}\right\}^{1/3}. \qquad (9)$$

In the light of dimensionless groups (7) and (8) equation (9) represents an asymptotic convection limit at which the influence of D on $\bar{q}$ disappears. There have been two special cases from which information on A_{GM} may be extracted. First, at low T, ($\rho_n/\rho \ll 1$), the lower limit of the Gorter–Mellink regime is one of many different "critical" velocities [13]. This limit allows evaluation of A_{GM} in terms of η_n. Secondly, at "high" temperature, $(T_\lambda - T)/T_\lambda$ at which $\rho_s/\rho \ll 1$, scaling relations have been proposed which permit a power law for A_{GM} as a function of (ρ_s/ρ).

Turning to the low temperature constraint we note that Dimotakis' criterion for the critical velocity $\bar{w}_c$ [14] expresses data on the basis of

$$\bar{w}_c D \rho A_{GM} = \mathrm{const} \approx 1/\pi. \qquad (10)$$

On the other hand, the critical Reynolds number for normal fluid is of the order 10^3 [13]

$$D \rho \bar{v}_n/\eta_n \sim 10^3. \qquad (11)$$

Comparing (10) and (11) we obtain $\eta_n A_{GM} = \mathrm{const}$ in the normal fluid depletion limit

$$\eta_n A_{GM} \sim \frac{1}{\pi \cdot 10^3} \approx K_{GM}^{-3}. \qquad (12)$$

In the vicinity of the lambda point the experimental data suggest that A_{GM} increases beyond all limits. For this case $(T_\lambda - T)/T_\lambda \ll 1$, Ahlers [15] has proposed a power law for A_{GM} with a constant exponent. As (ρ_s/ρ) is very small, one may expect that A_{GM} may be written in terms of $(\rho_s/\rho)^{-1}$ to some power. Data suggest that the ratio $N_q/(N_{\nabla P_T})^{1/3}$ is proportional to $(\rho_s/\rho)^{4/3}$ [16]. According to equation (9) this implies that in the "high" temperature limit we have

$$A_{GM} \sim 1/(\rho_s/\rho); \quad (\rho_n/\rho) \rightarrow 1. \qquad (13)$$

Comparing the two statements (12) and (13), we see that both constraints are satisfied when

$$A_{GM} = \mathrm{const}\,(\rho/\rho_s)/\eta_n \qquad (14)$$

(const $= K_{GM}^{-3}$). In an early outline [16] however, the data show considerable departures from equation (14) when D is very small. Therefore, we turn to recent experimental data and literature results for Gorter–

Mellink transport in the asymptotic limit under consideration, i.e. large D-values.

4. GENERALIZED CONVECTION EQUATION

The significance of equation (14) has been assessed by plotting the product $A_{GM}^{-1} \cdot \eta_n$, i.e. the dimensionless turbulent viscosity vs the superfluid density ratio. Figure 1 displays data for three different duct geometries. Annulus results obtained in the course of Taylor–Couette flow studies [17] constitute non-rotating limits of data sets for finite rotation rates [18]. Further, Fig. 1 includes slit results for He II between two parallel plates [19], and tube data [20]. Within data scatter the experimental information appears to be in reasonable agreement with equation (14) in the (ρ_s/ρ)-range covered. Therefore, we turn to a more complete inspection of data sets reported in the literature and obtained by us in the context of [17]. The resultant convection equation for axial heat flow is rearranged on the basis of equation (14) obtained for A_{GM} in the previous paragraph.

After insertion of equation (14) into equation (9) we rewrite q as

$$q = K_{GM}\rho_s ST\{(\rho_s/\rho_n)S|\nabla T|\eta_n/\rho\}^{1/3} \quad (15)$$

where K_{GM} is of the order of the reciprocal critical value of the product $A_{GM} \cdot \eta_n$, [equation (12)]. It is convenient to express equation (15) in dimensionless form on the basis of the grouping of variables in equations (7) and (8) for $L_c = D$

$$\frac{\rho \bar{w} D}{\eta_n} = K_{GM}\left\{\frac{D^3\rho|\nabla P_T|}{\eta_n^2}(\rho_s/\rho_n)\right\}^{1/3} \quad (16)$$

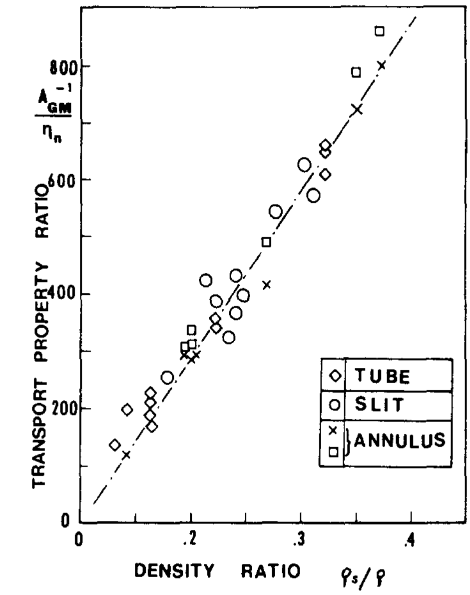

FIG. 1. Transport property ratio.

[with ∇P_T of equation (4)]. Figure 2 plots the dimensionless relative velocity (7) vs the group (8). Aside from the present data [17] (designated as R-12 to R-15), we have included results of [19–27]. The thermophysical properties have been evaluated at the arithmetic mean temperature of the He II in the duct; (symbols are listed in Table 2).

It is seen from Fig. 2 that within data scatter there is considerable support for equation (15), and (16) respectively. From a least square fit the (weighted)

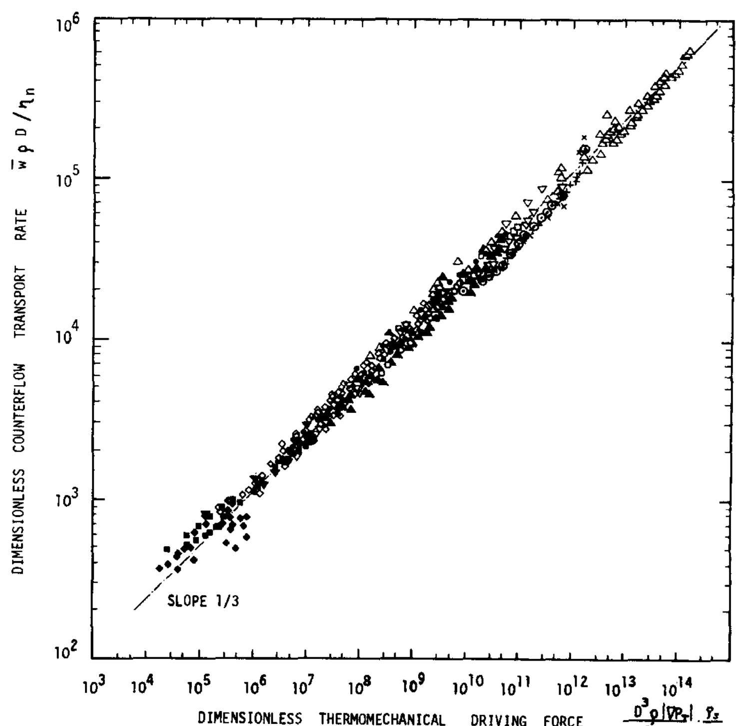

FIG. 2. Dimensionless counterflow rate vs dimensionless generalized driving force.

Table 2. Data symbols of Fig. 2

	Author(s)	Reference
◇	Chase	[20]
○	Linnet	[19]
■	Brewer–Edwards	[21]
◆	Childers–Tough	[22]
▽	Critchlow–Hemstreet	[23]
△	Eaton–Lee–Agee	[24]
▼	Keesom–Duyckaerts	[25]
▲	Keesom–Saris–Meyer	[26]
●	Vinen	[27]
	Soloski: Specimen	
+	R-12	[17]
×	R-13	[17]
⊙	R-14	[17]
□	R-15	[17]

average value of the constant is $K_{GM} = 11.3 \pm 1.4$. Systematic departures occur at low temperature gradients when the transition to Gorter–Mellink flow is initiated. This point has been discussed in detail elsewhere [22]. This implies that the numerical evaluation of the *critical* condition (12) leads to a somewhat different value of the constant. At the upper limit of ΔT the data scatter may be enhanced somewhat due to the inadequacy of the arithmetic mean temperature for property evaluation. Details of this effect and other possibilities have been addressed in [17]. The strong temperature dependence implies variable power law exponents $[d \log A_{GM}/d \log T]$ up to 3 and even higher. Because of the finite ΔT-range accessible by Gorter–Mellink convection, various sets of literature data may be compared in a meaningful way only when the T-dependence is considered (e.g. [28]).

Concerning the pressure influence on A_{GM}, we note that η_n-results are sparse at high pressures. From the few viscosity data available, we conclude that transport rates of [19] appear to agree with equation (16) in the range of pressure and temperature for which η_n is known. Also other recent data for pressurized He II [28–30] appear to be consistent with our final equation (16).

It is concluded that two disturbances exert only a minor influence on non-linear heat flow in He II at the present asymptotic Gorter–Mellink limit. One effect is the possibility of thermal shocks associated with second sound [8]. The other point is the possible occurrence of additional special forces, either in the superfluid or in the normal fluid. Thus, we find nearly "classical" behavior in the normal fluid depletion limit of He II which is quite compatible with expectation based on Newtonian fluid conditions. To prevent misunderstanding we emphasize that it is the large magnitude of the externally applied thermomechanical pressure difference which constitutes an extraordinary driving potential difference in He II systems.

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ANALYSE DIMENSIONNELLE ET EQUATION DU FLUX AXIAL DE CHALEUR DANS LA CONVECTION DE GORTER–MELLINK

Résumé—L'analyse dimensionnelle du transfert thermique dans des tubes remplis de He II est conduite sur la base des équations à deux fluides de Landau, pour la limite du fluide normal. Les relations fonctionnelles résultantes entre des groupes de variables, pour des fluides newtoniens, sont étendues jusqu'au point lambda (T_λ) en utilisant la dépendance à la température de la viscosité "turbulente" effective. L'équation de convection est en bon accord avec les données de nombreux auteurs et avec des résultats originaux de 1,2 K à T_λ et pour des diamètres de 10^{-2} à 1 cm.

AUFSTELLUNG EINER GLEICHUNG MITTELS DIMENSIONSANALYSE FÜR DEN AXIALEN WÄRMEFLUß VON HE II BEI GORTER-MELLINK-KONVEKTION

Zusammenfassung—Es wird eine Dimensionsanalyse des Wärmetransports durch mit He II gefüllte Leitungen auf der Grundlage der Zwei-Fluid-Gleichungen von Landau für den Grenzfall des Verschwindens des normalen Fluids durchgeführt. Resultierende Funktionen zwischen den relevanten Gruppen von Variablen für newtonsches Flüssigkeitsverhalten werden unter Verwendung der Temperaturabhängigkeit der effektiven 'turbulenten' Zähigkeit bis zum Lambda-Punkt erweitert. Die resultierende Konvektionsbeziehung befindet sich in guter Übereinstimmung mit Meßwerten vieler Autoren und unseren eigenen Resultaten von ungefähr 1, 2, K bis T_λ und für Durchmesser der Größenordnung von 10^{-2} bis 1 cm.

БЕЗРАЗМЕРНЫЙ АНАЛИЗ И УРАВНЕНИЕ ДЛЯ АКСИАЛЬНОГО ТЕПЛОВОГО ПОТОКА В СЛУЧАЕ КОНВЕКЦИИ ГОРТЕРА-МЕЛЛИНКА (HeII)

Аннотация — Выполнен размерный анализ переноса тепла в заполненных гелием-II каналах на основании уравнений Ландау для двух жидкостей. Функциональные соотношения между соответствующими группами переменных, описывающих поведение ньютоновской жидкости, обобщены до лямбда-точки (T_λ) с помощью температурной зависимости эффективной «турбулентной» вязкости. Полученное уравнение конвекции хорошо согласуется с экспериментальными данными многих авторов, а также с результатами настоящей работы для диапазона температур от 1,2 К до T_λ в каналах диаметром от 10^{-2} до 1 см.